Solution for HW6 10-11-2016

§49) 1) b) Since $f(z) = ze^{-z}$ is entire, by Cauchy-Goursat thm, $\int_C f(z)\,dz = 0$.

c) Note that $z^2+2z+2=0$

$$\Leftrightarrow \quad z = \frac{-2 \pm \sqrt{4-4(2)}}{2}$$

$$= -1 \pm 2i,$$

which lie outside the region $\{z \mid |z| \le 1\}$.

Hence, $f(z) = \dfrac{1}{z^2+2z+2}$ is analytic on $\{z \mid |z| \le 1\}$.

By Cauchy-Goursat thm, $\int_C f(z)\,dz = 0$.

d) Recall that $\cosh z = 0 \Leftrightarrow z = \left(\frac{\pi}{2} + n\pi\right)i,\ n \in \mathbb{Z}$, which lie outside the region $\{z \mid |z| \le 1\}$.

Hence, $f(z) = \operatorname{sech} z$ is analytic on $\{z \mid |z| \le 1\}$.

By Cauchy-Goursat thm, $\int_C f(z)\,dz = 0$.

2) Let D be the region bounded by C_1 and C_2.

a) Note that $3z^2+1=0 \Leftrightarrow z = \frac{\pm 1}{\sqrt{3}} i \notin D$

Hence $f(z) = \dfrac{1}{3z^2+1}$ is analytic on D.

So we have $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

b) Recall that $\sin\frac{z}{2} = 0 \Leftrightarrow \frac{z}{2} = n\pi,\ n \in \mathbb{Z}$

$\Leftrightarrow z = 2n\pi,\ n \in \mathbb{Z}$,

which lie outside D.

Hence $f(z) = \dfrac{z+2}{\sin(\frac{z}{2})}$ is analytic on D.

$$\text{So } \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

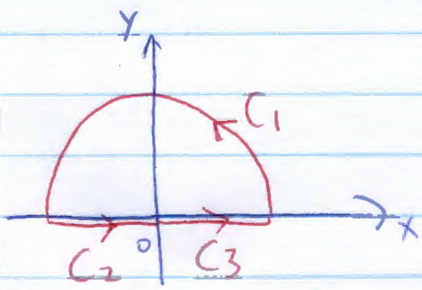
c) Note that $1-e^{z}=0 \iff e^{z}=1$
$\iff z = 2n\pi i \,, n\in\mathbb{Z}$,
which lie outside D.

Hence $f(z)=\dfrac{z}{1-e^{z}}$ is analytic on D.

$$\text{So } \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

3) Let $C_{100}=\{z \mid |z-(2+i)|=100\}$
Let D be the region between C_{100} and C.
Since the function $f(z)=(z-2-i)^{n-1}$ is analytic
on $\mathbb{C}\setminus\{2+i\}$, in particular, it is analytic on D.
So
$$\int_{C} f(z)\,dz = \int_{C_{100}} f(z)\,dz$$
$$= \begin{cases} 0 & , n=\pm1,\pm2,\dots \\ 2\pi i & , n=0. \end{cases}$$

6) On C_1,
$$\int_{C_1} f(z)\,dz = \int_0^{\pi} e^{\frac{i\theta}{2}}\cdot ie^{i\theta}\,d\theta$$
$$= i\int_0^{\pi} e^{\frac{i3\theta}{2}}\,d\theta$$
$$= i\,\frac{2}{3i}\,e^{i(\frac{3\theta}{2})}\Big|_0^{\pi}$$
$$= -\frac{2}{3}(1+i)$$



On C_2, $\displaystyle\int_{C_2} f(z)\,dz = \int_{-1}^{0} \sqrt{|x|}\, e^{\frac{i\pi}{2}}\, dx$

$$= -i \int_{-1}^{0} \sqrt{-x}\,(d(-x))$$

$$= -i \int_{1}^{0} \sqrt{y}\, dy$$

$$= -i\, \frac{2}{3} y^{\frac{3}{2}} \Big|_{1}^{0}$$

$$= \frac{2}{3} i$$

On C_3, $\displaystyle\int_{C_3} f(z)\,dz = \int_{0}^{1} \sqrt{x}\, dx$

$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_{0}^{1}$$

$$= \frac{2}{3}$$

Hence, $\displaystyle\int_C f(z)\,dz = \int_{C_1+C_2+C_3} f(z)\,dz = 0$

Note that when $\theta = 0$,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\sqrt{r}-0}{r} = \lim_{r\to 0} \frac{1}{\sqrt{r}} = \infty$$

Hence $f(z)$ is not differentiable at $z=0$.
So Cauchy-Goursat thm does not apply.

§52] 1) b) By CIF,

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = \int_C \frac{\cos z/(z^2+8)}{z}\,dz$$

$$= 2\pi i\, \frac{\cos(0)}{0^2+8}$$

$$= \frac{\pi i}{4}$$

c) By CIF, $\int_C \frac{z\,dz}{2z+1} = \frac{1}{2}\int \frac{z}{z-(\frac{-1}{2})}dz$

$= 2\pi i \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)$

$= -\frac{\pi i}{2}$.

d) By CIF, $\int_C \frac{\cosh z}{z^4}dz = 2\pi i\ \left(\cosh^{(3)}(0)\right)$

$= 2\pi i\ \sinh(0)$

$= 0$.

2) a) $\int_C g(z)\,dz = \int_C \frac{1}{z^2+4}dz$

$= \int_C \frac{1}{(z+2i)(z-2i)}dz$

$= \int_C \frac{1/(z+2i)}{z-2i}dz$

$= 2\pi i\left(\frac{1}{2i+2i}\right)$

$= \frac{\pi}{2}$

b) $\int_C g(z)\,dz = \int_C \frac{1/(z+2i)^2}{(z-2i)^2}dz$

$= (2\pi i)\frac{d}{dz}\left(\frac{1}{z+2i}\right)^2\Big|_{z=2i}$

$= 2\pi i\ \frac{-2}{(2i+2i)^3}$

$= \frac{\pi}{16}$

3) $g(2) = \int_C \frac{2s^2-s-2}{s-2}\,ds$

$= 2\pi i\,(2(2)^2-2-2)$ (by CIF)

$= 8\pi i$

When $|z|>3$, the function inside the integral is analytic on $\{s : |s| \le 3\}$. So $g(z)=0$ by Cauchy Goursat thm.

7) By CIF, $\int_C \frac{e^{az}}{z}\,dz = 2\pi i\, e^{(0)} = 2\pi i$.

On the other hand,

$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}}\, ie^{i\theta}\,d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta} \cdot e^{ia\sin\theta}\,d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)d\theta$$

$$= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta$$

By comparing the imaginary part on both sides,

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi$$

Since $\int_{-\pi}^{0} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \int_{\pi}^{0} e^{a\cos(-\phi)}\cos(a\sin(-\phi))(-d\phi)$

$$= \int_0^{\pi} e^{a\cos\phi}\cos(a\sin\phi)\,d\phi,$$

where $\phi = -\theta$, we have

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$$